Option pricing

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Notation we use this Chapter will be as follows:

S_o	: Price of the share at time 0
S_T	: Price of the share at time T
T	: time to maturity of the option
r	: risk free rate of interest
X	: strike price of the option
c	: value of a European call option per share
p	: value of European put option per share

Bounds of value for option prices:

Upper and lower bounds for call options:

The payoff of a call option is Max(S-X,0). That is to say, if the current prevailing price of the asset is \$ 15, and the strike price is \$ 10, the value of the call option is \$ 10. The call option is worthless if the value of the asset is \$ 10 or less.

Quite clearly, the value of the option is directly variable with the price of the asset. As the asset price goes up, the value of the option goes up; as the price of the asset goes down, the value of the option goes down, but never declining below zero.

However, since the option is being valued at point 0, though expiring at point T (thinking of a European option), the value of the asset is surely not determinate at point 0. Hence, the value of the option is the expected value of the asset at T, minus the strike price.

$$c = E(S_T) - X \qquad (1)$$

On reflection, we may see that the maximum value of p cannot be more than the discounted value of X, discounted at risk free rate. That is,

$$c \leq X.e^{-rt} \qquad (2)$$

Why is that so? That is because the value of an asset (specifically a share in a limited liability company) can never fall below zero. If the value of the call option is more than $X.e^{-rt}$, I can then sell the option, and invest the proceeds at a risk free rate, such that when compounded to time T, the value in hand becomes more than X, and since I hold the option to buy the asset at X, I am actually buying it for less than zero. As this does not make sense, hence the upper bound for value of an option is as shown in Eq. (2) above.

What is the lower bound for the option value? The lower bound for a European call option is given by the following:

$$c \geq S_o - X.e^{-rt} \qquad (3)$$

How do we explain the lower limit? Note that the option price reflects the difference between the future expected value of the share and the strike price – Eq (1) above. The future expected value of the share can never be less than $S_o\, e^{rt}$. This is because if the future expected value is less than $S_o\, e^{rt}$, it is possible to short-sell the share, invest the proceeds at risk free rate, and bag a profit at time T. Since the future expected value of a share cannot be less than current price compounded at risk free rate, therefore, the value of the option cannot be less than (current prevailing price – strike price discounted at risk free rate). If that is so, one would buy a call option, short-sell a share, invest the net proceeds at risk free rate, and be left with a profit no matter whether the future price of the share goes up or come down.

Example 1

Imagine the following situation:

$S_o = 100$
X = 102
Risk free rate = 5%
Prevailing price of the call option = 2
Expiry= 1 year

This is inefficient, as it leads to arbitrage profits. We can see this by observation. At 5% risk free rate, the compounded value of the current price of the share is 105.12. As the strike price is 102, there is a risk-free gain of 3.12 at time T that is inherent in the transaction. If this gain is discounted to present, the option must have a least value of 2.9746. That is the minimum value of the option. Putting the values in equation (3), we have

$$c \geq 100 - 102/\exp(5\%)$$
$$=2.9746$$

In the present example, a trader may buy call option at \$2, short sell the share at \$ 100, thus getting cash of \$ 98. Invested at risk free rate, this cash becomes 103.02. If, upon maturity, the price of the share is more than \$ 102, the trader exercises the option and buys the stock at \$ 102, thereby making a profit of 103.02 – 102. If the stock is less than 102, the trader makes a gain on the short position as well.

Upper and lower bounds for put options:

The value of a put option

$$p= X - E(S_T) \qquad (4)$$

In the worst situation of price of an asset, the price can go to zero. Hence, the value of the option cannot be higher than discounted value of the strike price. That is to say,

$$p \leq X.e^{-rt} \qquad (5)$$

Let us now look at the lower bound for value of the European put option. As we said before, the future expected price of a share cannot be less than the compounded value of its current priced, compounded at risk free rate. To put it differently, the present prevailing price of a share cannot be less than the discounted value of future expected price, or discounted value of the strike price. If it is, the same should be reflected in the price of the put option. That is to say

$$p \geq X.e^{-rt} - S_o \qquad (6)$$

Example 2
Imagine the following situation:

$S_o = 100$
X = 110
Risk free rate = 5%
Prevailing price of the put option = 4
Expiry= 1 year

This is also inefficient, as there is an arbitrage opportunity. If I buy the share, and buy the call option, thereby investing 104, borrowing money at risk free rate[1], I need to return 109.33. Because of the call option, I have anyway locked my selling price to 110 – hence, I am assured of a risk free profit of 110-109.33. If the share price is more than 110, I make profit on the difference between the prevailing price and 109.33. Hence, the minimum value of the option should be (discounted value of 110 minus the present prevailing price).

Put-call Parity:

Both put and call options take a position on the prices of securities. Hence, there must be a relationship between these. Let us examine the following:

$$c = \max(S_T - X, 0) \qquad (7)$$
$$p = \max(X - S_T, 0) \qquad (8)$$

If, to equation (7), I add cash equal to $X.e^{-rT}$ today, which I invest at r rate of return, upon maturity, equation (7) gives the following:

$$c + X.e^{-rT} \text{ matures to } \max(S_T - X + X, X)$$
or
$$c + X.e^{-rT} \text{ matures to } \max(S_T, X) \qquad (9)$$

If to equation (8), I add long position in the share, I have the following

[1] Why risk free? The position is riskless, as illustrated in the example.

$$p + S_o \text{ matures to max } (X - S_T + S_T, S_T)$$

or $$p + S_o \text{ matures to max } (X, S_T) \qquad (10)$$

Equation (10) is purely intuitive – all it means is that if I am holding a long position in an asset, and also an option to put it, then the value of my wealth is higher of the actual price of the asset on maturity, or the strike price.

Since Equation (9) and (10) both have equal values, we can also write them as follows:

$$c + X.e\text{-}^{rT} = p + S_o \qquad (11)$$

This relationship is called *put-call parity*. The put-call parity establishes the relationship between put and call prices of a share, with the same strike price and same maturity. That is to say, from the put prices, the call prices can be deduced, and vice versa. From (11), we have

$$c = p + S_o - X.e\text{-}^{rT} \qquad (12)$$

And

$$p = c - S_o + X.e\text{-}^{rT} \qquad (13)$$

Example 3
Let us suppose we have the following information:

$S_o = 100$
$X = 110$
Risk free rate = 5%
Prevailing price of the put option = 6
Expiry= 1 year

Putting these values in Eq. (12), the price of the call option cannot be more or less than $(6+100 - 110/e^{.05}) = 1.36$. At exactly 1.36, it creates a situation of no-profit/no-loss if I buy a call option, short the put and short the share. If I do so, my upfront cashflow is -1.36+ 100 +6 = 104.63. If this money is invested at risk free rate of 5%, it exactly compounds to 110, which is the price of the share that is locked (locked due to combination of the long call and short put). If the prevailing call price is less than 1.36, I can make a risk free profit. If the prevailing call option price is more than 1.36, then I can short the call, long the put and long the share, and lock a risk free profit.

Valuation of options

The discussion we had above allowed us to see upper or lower limits on value of options. However, we have still not been able to compute the value of options. Needless to reiterate, the value of the option itself depends on the predicted value of asset prices upon maturity (once again, considering European options). However, for given set of predicted values, is it possible to compute the value of the option? In this section, we build the background based on which binomial or Black-Scholes valuation of options is done.

Setting up a riskless portfolio:

Example 4

Supposing we consider the following:

$S_o = 100$
X for call option = 110
Possible increased price upon maturity: 120
Possible reduced price upon maturity : 90
Risk free rate = 5%
Maturity = 1 year

In this example, we are considering only 2 possible scenarios – an increased price and a reduced price. We are considering a call option: hence, the option will have a value of 10 if the price is 120. The option will have zero value in the other situation of price being 90. If these are the only 2 probabilities, is it possible to construct a riskless portfolio[2] consisting of Δ shares for every 1 option? When is the portfolio riskless? When the value of the portfolio is the same in both the possible outcomes.

Let us assume that it is possible to equate the value in both the up and the down situation. So, I long Δ shares and short 1 call option. In the situation of the price being 120 on maturity, the option gives a value of 10, and the long position gives the value of the stock. Hence, the total value of the portfolio is $(120\Delta - 10)$.. In case the price is 90, the option has zero value, and the long position has value equal to 90Δ. As we assumed both these values are the same, therefore:

$$120\Delta - 10 = 90\Delta. \tag{14}$$

Hence, the Δ equals to 0.33. That is to say, a portfolio consisting of 33% of the shares for which I sell call options will put me in a risk-free position. We may check this for the above example. If I long 30 shares, and short call option on 100 shares, upon maturity, the value of the portfolio is either (1200*33- 10*100) = 3000, or 33*90= 3000.

This would mean, if the likely outcomes of the share prices are known as we had assumed, then a portfolio consisting of long position in 33 shares, and short position in 100 call options, will be risk-free, and would have a value of 3000 upon maturity.

From this, we can also compute the value of the portfolio today. As the portfolio is risk-free, we can compute its discounted value at risk free rate of return, that is, $3000/e^{.05}$ = 2853.69.

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[2] Here, portfolio means the shae and the option

Option values from riskfree value of the portfolio:

From this, it is also possible to deduce the value of the option. Note that we are going long on 33 shares and shorting 100 call options (on which we get money equal to c per option). Hence, the cost of the portfolio today would be

$$33.3333*100 - c*100 = 2853.69 \qquad (15)$$

This value must be equal to the present value of the maturity value of the portfolio, computed above as 2853.69. Hence,

$$c = 4.7964$$

Binomial valuation of options

Probabilities of upward and downward movements:

If the price of a share at time $S_o = 100$, what is the probability that at time T=1, the price is either 120, or 90? We have discussed earlier that the price of a share must at least appreciate at risk free rate. Assume the risk free rate is 5%. At the end of 1 year, 100 compounds to 105.1271. Hence, the probabilities of the value at time T being higher (S_u) or lower (S_d) are given by the following:

$$\text{Probability (p) of value being } S_u = \frac{S_o * e^{rT} - S_d}{S_u - S_d} \qquad (16)$$

This formula simply implies that if the higher value being considered is upto $S_o * e^{rT}$, the probability is 1. As the higher value exceeds $S_o * e^{rT}$, the probability will keep coming down linearly as the higher value increases. Of course, the probability of the value being S_d is simply (1-p).

In our example, the higher price was 120, and the lower price was 90. Hence, the probability of the price being 120 is 0.504237 and that of the value being 90 is 0.495763.

Valuation of option using probabilities:

If, as we did in the example above, we assume only 2 possible prices of the security at time T, and we know their respective probabilities, then, computing the value of the option is a simple case of expected value computation. In our example, the strike price of the call option was 110. If the price at time T is 120, the option pays 10. If the price is 90, the option pays nothing. Hence, the expected value of the option upon maturity is:

$$10*0.504237 = 5.04. \qquad (17)$$

This is the expected value of the option upon maturity. Hence, its discounted value today will be $5.04/e^{rT}$. This comes exactly to the value 4.796451 that we computed in Equation (15).

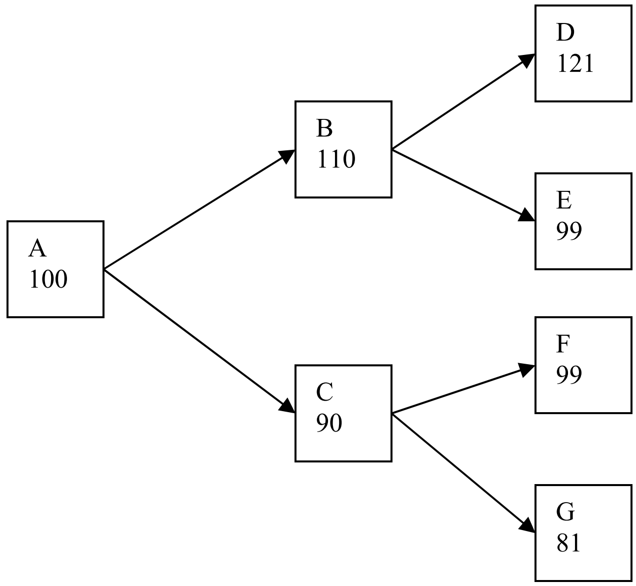
The process of computing option value two possible outcomes of the share prices is called binomial valuation.

Two step binomial valuation:

The branching out of the probabilities from S_o to S_u and S_u cannot remain limited to just one movement. In fact, infinite number of such ups and downs may happen before the maturity of the option. To understand the methodology and the computational rigor involved in the binomial valuation, let us consider a second level of branching. As we build two or more levels of branching, we are constructing a *binomial tree*. Let us make the following assumptions:

S_o = 100
X for call option = 110
First movement in prices: 3 months
Possible increase: 10%
Possible decrease: 10%
Second movement in prices (from end of first movement): 3 months
Possible increase: 10%
Possible decrease: 10%
Risk free rate = 5%
Maturity = 6 months

At the first movement, the price may accordingly either be 110, or it may be 90. First upward movement having taken place, the price may either be 121, or it may be 99. The first downward movement having taken place, the price may either be 99, or it may be 81. The scenarios are reflected in the following tree:



To find the value of the option at node A, we first need to find the values at nodes B and C. At node B, there are two scenarios – the price of the share rising upto 121, or falling to 99. In case it rises to 121, the option gives a value of 11; or else, it is worthless. Let us assign probabilities of scenarios D and E at node B. Applying the formula in Equation (16) (note that the maturity is only 3 months – hence, risk free rate has to be compounded for 3 months only), we have:

$$p = (110*e^{5\%*.25} - 99)/(121-99) = .5629$$
$$(1-p) = .4371$$

Hence, the expected value of the option is (11*.5629)+(0*.4371) =6.1918

This is the maturity value of the option at the end of 6 months – to compute the value at node B, we have to discount it for 3 months. Hence, the present value of the option at node B is $6.1918/ e^{5\%*.25} = 6.1149$.

Value of the option at node C is clearly zero, as in both the points F and G, the option has a zero value.

We can now value the option at the starting point, viz., A. We may once again compute the probabilities here, as follows:

$$p = (100*e^{5\%*.25} - 90)/(110-90) = .5629$$
$$(1-p) = .4371$$

Let us not get surprised as to how we get the same probabilities – it is because the percentage increase and decrease are the same, and the maturity is also the same. Since we know that the value of option at B is 6.1149, and at C, it is zero, applying the probabilities, the expected value becomes (6.1149*.5629)+(0*.4371) =3.44208. Once again, this is discounted back to point A. Hence, the value of the option at point A is 3.399319.

Valuation of put options:

For valuing put options, we can use two approaches – the expected value approach, as also computing the value of the riskless portfolio with Δ shares as was done in case of call option. A third possible option is, since we have computed the value of the call already, we can apply the same value into the put-call parity formula and obtain the value of the put.

Using the expected value approach:

The expected-value-based valuation of call options will be applicable to valuation of call options too. That is to say, probabilities may be assigned to the upward and downward movement using the same formula as we have used in Eq. (16). Let us take an example to understand the valuation.

Example

S_o = 100
X for put option = 110
Possible increased price upon maturity: 120
Possible reduced price upon maturity : 90
Risk free rate = 5%
Maturity = 1 year

Step 1: Payoffs
In case of the price going up, the option has zero value. In case of price going down to 90, the option has a value of 20.

Step 2: Probabilities
Using the formula in Eq (16), the probability of price going up us computed as follows:

$p = (100*e^{5\%} – 90)/(120-90) = 0.50424$
$(1-p) = 0.495763$

Step 3: Expected value of the option
Hence, the expected value of the option is (0*0.50424)+(20*0.495763) = 9.91526

Step 4: Present value of the option
We may now compute the present value at 5% rate for 1 year = 9.431687

Using the call option value:

We may also compute the value of the option from the value of the put option with the same strike rate and same maturity. Earlier, in Example 4, solving eq. (15), we have computed the value of the call option for the same example. The value came to 4.7964.

We can compute the value of the put option applying the put-call parity Eq. (13) as follows:

$$p = c - S_o + X.e^{-rT} \qquad (13)$$

Hence

$$p = 4.7964-100+120/e^{.05} = 9.4316$$

Using the risk-free portfolio approach:

There may be yet another approach to valuing the put option using the risk-less portfolio approach. If I long a put, I make a gain in case of downward movement in shares. If I also long shares, I make a gain in case of upward movement of prices. Suppose I long options on 1 share, and create a long position on Δ shares. In the case of the price going up to 120, the value of my portfolio is value of the long position in shares (120 Δ) and the option is worthless. In case the price declines to 90, the put option gives me a pay off of 20, and the shares are worth 90 Δ. Assuming with Δ shares, these two positions are the same, we have

$$120\,\Delta = 90\,\Delta + 20$$

Hence, value of Δ comes to 2/3.

Applying the above value to either side of the equation above, the value comes to 80. When we discount it back to present at 5% risk free rate, the value of the portfolio on day 0 comes to 76.09835.

However, the actual cost of buying 2/3 shares is 200/3. Besides this, I would have longed the put option. Hence, the actual cost of creating the portfolio is

$$200/3+p= 76.09835$$

Solving this, we get the value of the put as 9.431687 – the same that we got by the other two approaches.

Black Scholes option pricing formula

From binomial to normal:

The valuation of the option in the binomial expansion was based on two outcomes on every successive change in stock prices. In reality, stock prices change momentarily. Hence, there are near infinite movements in stock prices over the term of the option. That would there would be infinitely large number of nodes if we were to apply the binomial expansion technique.

As simple way of achieving this is to apply normal distribution instead. As the number of successive occurrences becomes very large, and the price changes become small, the distribution of price changes tends to be normal. In case of stock prices, however, normal distribution is not applicable, as a standard normal distribution assumes positive and negative values scattered along a mean of 0. Stock prices cannot be negative – hence, stock prices do not follow normal distribution. Instead, changes in stock prices do – hence, stock prices are taken to be normal on the log scale – that is, logs of stock prices are normally distributed.

The Black Scholes formula

Applying properties of geometric Brownian motion and solving partial differential equations thereof, Black, Scholes and Merton provided a formula for option valuation which has since completely changed the way the world looks at option valuation. The formula is applicable to European call and put options on non-dividend paying stocks. The value of put and call options as per Black Scholes is given by the following:

$$c = S_0N(d_1) – Xe^{-rT}.N(d_2)$$
$$p= Xe^{-rT}.N(-d_2)- S_0N(-d_1)$$

where
$d_1 = (\ln(S_o/X)+(r+\sigma^2/2)T)/\sigma\sqrt{T}$
$d_2 = d_1-\sigma\sqrt{T}$
σ is the standard deviation in the natural logs of the asset values

What does Black Scholes mean?

The two parts of the BS equation relate to the hedge ratio, and the discounted value of the expected value of the pay-off from the option where the prevailing price is less than the strike price. The first part, $N(d_1)$ represents the number of shares that needs to be held long to create the riskless portfolio. The second part is amount of money that needs to be in hand today to create a riskless position. The difference between the two values is the value of the call option.