Understanding duration and convexity of fixed income securities

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Notation

y : yield

p: price of the bond

T: total maturity of the bond

t: any given time during T

C_t: Cashflow from the bond at time t

D_m: McCauley duration of the bond

D^*_m: Modified duration of the bond

V: convexity of the bond

What are fixed income securities – securities that carry a fixed rate of interest or coupon rate, or a fixed redemption value with or without a coupon. Examples may be – treasuries or dated government securities, coupon bearing corporate bonds, zero coupon corporate bonds, certificates of deposit, commercial paper, etc. It is not necessary that a fixed income security will have a coupon – for example, a zero coupon bond does not have a coupon rate, and nevertheless, it gives rise to a fixed income as its issue/purchase price and redemption value are determinate. In short, any financial instrument which gives to the investor a fixed rate of return if the instrument is held to maturity is a fixed income instrument.

There is a basic difference between the prices of equities and fixed income securities – in case of equities, the prices are based on expected earning and expected residual value of the issuer; in case of fixed income securities, the cashflows promised to the bondholder are fixed – hence, expected earnings will not make a difference to the bondholder.

This does not mean that there is no volatility in prices of fixed income securities – the prices of fixed income securities change primarily due to change in investors' yield expectation. The yield expectation, in turn, may be seen as composed of (a) a risk free rate; (b) a risk premium or spread for the risk(s) inherent in holding the security.

Quick understanding of bond types:

In terms of callability feature, bonds may be **callable** or **non-callable**. Callable bond is one which can be prepaid by the bond issuer prior to its maturity. [**Teaser question:** When will a bond issuer elect to call a bond?]

In terms of convertibility, a bond may be **convertible** or **non-convertible**. Convertible bonds contain an embedded option – the option to convert is akin to a call option on the equity, and therefore, become a hybrid between fixed income and equity securities.

From the viewpoint of redemption features, a bond may be **bullet bond**, **amortising bond**, and **zero coupon bond**. Bullet bonds pay regular coupons and repay the entire principal upon maturity. Amortising bonds repay principal over the tenure of the bond – may be either in form of equal repayment of principal, or equal instalments of principal plus interest, or repayment of principal in stated instalments, after a certain number of years. Zero coupon bonds do not carry any coupon –the difference between the issue price and the redemption value is the return of the investor. The redemption price is normally the face value of the bond, and the issue price is the discounted value – hence, they are also called **deep discount bonds**.

The price of a fixed income security:

Let us assume the risk free rate of return is flat, that is, the same over different maturities. Let us, likewise, assume that risk premium or spread is also fixed over different maturities. That would mean investors' expected yield (y) can be expressed as a constant across different time periods. If so, the price of the fixed income security, say bond, will be the present value of the cashflows of the bond over different points of time (t). We can state the price of the bond as:

$$P = \sum_{t=1}^{n} \frac{C_t}{(1+y)^t}$$

[Duration 1]

Example 1:

Let us assume we have a bond giving a coupon of 8%, say, payable yearly, and paying face value of $ 1000 on maturity of 5 years. If investor's expected yield is 7%, what is the price of the bond?

We compute the present value of the bond cashflows at the yield rate. Computation is shown below:

Yield		7%
Period	cashflows	
1	80	74.76636
2	80	69.8751
3	80	65.30383
4	80	61.03162
5	1080	770.0251
Price of the bond		1041.002

Bond coupons are conventionally payable either half-yearly or quarterly. We take one more example to illustrate periodic bond payments:

Example 2:

Let us assume we have a bond giving a coupon of 8%, say, payable half-yearly, and paying face value of $ 1000, along with a redemption premium of 5% on maturity of 5 years. If investor's expected yield is 7%, what is the price of the bond?

We compute the present value of the bond cashflows at the yield rate. Computation is shown below:

Yield		7%
Half year	cashflows	
1	40	38.64734
2	40	37.34043
3	40	36.07771
4	40	34.85769
5	40	33.67893
6	40	32.54003
7	40	31.43964
8	40	30.37646
9	40	29.34924
10	1090	772.7215
Price of the bond		1077.029

Note that the last cash inflow in the example above includes the redemption premium.

Bond yields

If a bond is bought at par, and is redeemable at par, then the coupon rate on the bond is the explicit rate of return – this is rate is also the implicit rate of return on the bond. However, assuming there is a secondary market in bonds, a fixed income investor may buy the bond at more than the face value or less than the face value, or a bond may be redeemable at a premium or discount. Hence, the rate of return on a bond may differ from the coupon rate.

The simplest computation is to compute the **current coupon** of the bond, that is, the coupon on its purchase price. For example, if a bond giving a coupon rate of 8% on a face value of $ 1000, maturing after 5 years, is available at a price of $ 950, the current coupon rate is 80/950 = 8.42%.

Surely enough, the current coupon is only a thumb-rule computation and does not give an idea of the real rate of return. So, the real rate of return is the YTM, that is **yield to maturity** of the bond. The YTM is nothing but internal rate of return of the bond. In the case above, the YTM is computed thus:

Time	Cashflows
0	-950
1	80
2	80
3	80
4	80
5	1080
	9.30%

The relation between the current coupon rate and the YTM is like this:

- If the current coupon rate is more than the coupon rate, YTM is more than the current coupon rate.
- The gap between the current coupon rate and the YTM will be a function of the maturity of the bond – the longer the maturity, the gap will be lesser.

Price-yield relationship

If the yield goes up, the discounted values of the bond inflows will come down – hence, the price of the bond declines. That is to say, fixed income securities lose in value if there is an increase in (a) risk free rates; and/or (b) risk premium due to perceived increase in probability of default or other factors. Thus, thus, bond prices are a function of yields, and the relationship is inverse.

We take an example below to understand the inverse relation between yields and bond prices.

Example 3

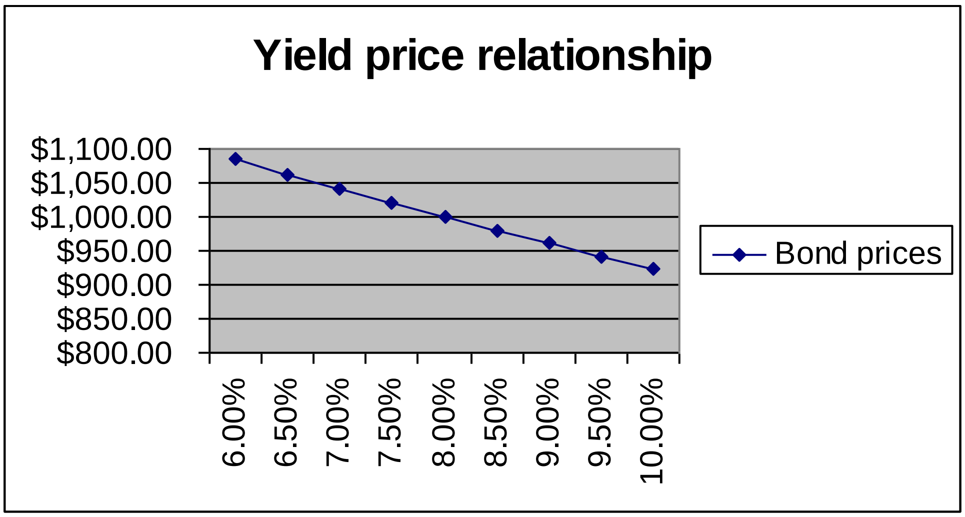
Let us take the bond in Example 1, paying a coupon of 8% over 5 years, repaying bullet on maturity. If investors yields are from 6% to say 10%, incrementing by 0.5%, what are the bond prices for each such yield?

Period	cashflows	Yields	Price	Resulting decrease
1	80	6.00%	$1,084.25	
2	80	6.50%	$1,062.34	$21.91
3	80	7.00%	$1,041.00	$21.33
4	80	7.50%	$1,020.23	$20.77
5	1080	8.00%	$1,000.00	$20.23
		8.50%	$980.30	$19.70
		9.00%	$961.10	$19.19
		9.50%	$942.40	$18.70
		10.00%	$924.18	$18.22

As we may notice, with increasing yields, bond prices have come down. However, we may also notice that while the increase in the yield is 0.5% in each step, the decrease in bond prices is not constant – the amount of decrease is decreasing.

That is to say:

- With increase in yields, bond prices come down; with decrease in yields, bond prices go up.
- However, with increasing yields, bonds prices decrease at a decreasing rate; with decreasing yields, bond prices increase at increasing rate.
- Hence, the relation between yields and bond prices is not linear - it is curvilinear, convex from below.



The graph below shows the yield-price relationship – however, one may be visually seeing a straight-line. The line, in fact, is not straight, as clearly shown by the decrements in Example 3 above. The extent of convexity depends on several factors that we discuss later.

Price sensitivity of bonds:

As in case of equities, the investors in fixed income securities is concerned about the volatility of bond prices in relation to yields. As we can see in the graph above, interest rates or yields are the affecting variable, on which prices of bonds get affected. If the yield (y) changes by a certain amount (Δy), then the resulting increase/decrease in the price (p) of the bond (Δp) will be given by the slope of the curve that establishes this relationship. In bond parlance, the sensitivity of bond prices to yields is captured by *duration* of the bond. Why is the sensitivity referred to as duration and not the beta of the bond or similar name? This, perhaps, has to do with the history of fixed income investments, when investors used to approximate the sensitivity by looking at the weighted average time for which money is invested in a bond – the longer this weighted average time or duration, the more is the sensitivity of the bond price to yield. In today's parlance, there is not much reason for bond investors to be relying on approximate measures as the discounted values for any given change in yields can easily be computed on a computational tool. However, bond investors have been customarily used to using duration (and convexity, as discussed below) and quick and handy tools for understanding the sensitivity of bond prices to yields.

One of the most common computations of duration is McCauley duration (D_m) is simply the present value of each cashflow, weighted by the timing of the cashflow, divided by the price of the bond, that is to say:

$$D = \sum_{t=1}^{n} \frac{C_t . t}{(1+y)^t}$$

We can visualize the present value of each cashflow as money invested the bond, as the present value is the value of the cash inflow today. Each such present value remains invested upto its extraction time, that is, the timing of the cash flow. Hence, if we take the sum of the product of the present value of each cashflow and their respective time, and divide such sum by the sum of the present values (which is also the price of the bond), the result is the weighted average time for which the price of the bond was invested.

Example 4

Let us take a simple example. Assume a bond gives a coupon rate of 8%, payable annually, on a face value of $ 1000, maturing after 5 years, and is available at a price of $ 950. Since we have done this example before, we know that the YTM of the bond is 9.30%. Let us compute the duration of the bond:

Computation of duration

Time		Cashflows	Present value (PV)	PV weighted by time	
1		**2**	**3**	**4**	
0		-950			
1		80	73.19618	73.19618	
2		80	66.971	133.942	
3		80	61.27527	183.8258	
4		80	56.06394	224.2558	
5		1080	692.4936	3462.468	
	YTM	9.30%	950	4077.688	
	Duration			4.292303	years

Col 4 above is the product of col 3 and col 1. The last number in Col 3 is the sum of the present values, or the price of the bond. The last-but-one number in Col 4 is the sum of the products given in Col 4, and the last number in Col 4 is obtained by dividing 4077.68 by 950. That is the duration expressed in years.

Example 5

Let us take the same example, but with half-yearly cashflows. Assume the bond gives a coupon rate of 8%, payable semi-annually, on a face value of $ 1000, maturing after 5 years, and is available at a price of $ 950. We will first have to compute the YTM of the bond which we do below. We the go about computing the duration of the bond as follows:

Computation of duration

Time (half years)	Cashflows	Present value (PV)	PV weighted by time	
1	**2**	**3**	**4**	
0	-950			
1	40	39.09378	39.09378	
2	40	38.20809	76.41618	
3	40	37.34247	112.0274	
4	40	36.49646	145.9858	
5	40	35.66961	178.3481	
6	40	34.8615	209.169	
7	40	34.0717	238.5019	
8	40	33.29979	266.3983	
9	40	32.54536	292.9083	
10	1040	827.0088	8270.088	
YTM (half yearly)	4.64%	1148.598	9828.937	
YTM (yearly)	9.27%	Duration	8.557337	half-years
			4.278669	years

Modified duration:

The McCauley duration is only a measure of the weighted average time for which investment in the bond is outstanding – it is not an indication of the rate of change in yhr price of the bond with a given change in the yield.

The rate of change in price, expressed as a function of price, can be obtained by the slope of the price/yield curve, that is, by the first differential of price/yield relationship. Recall equation Duration 1:

$$D = \sum_{t=1}^{n} \frac{C_t . t}{(1+y)^t}$$

Now, differentiating p with respect to y, we have

$$\frac{\partial P}{\partial y} = -\sum_{t=1}^{n} \frac{C_t . t}{(1+y)^{t+1}}$$

[Duration 2]

This is called the **modified duration** of the bond (D^*_m).

This may also be written as:

$$D^*_m = \frac{\partial P}{\partial y} = -\frac{D_m}{(1+y)}$$ **[Duration 3]**

Therefore, the relation between McCauley duration and modified duration is as follows:

$D^*_m = -D_m / (1+y)$ **[Duration 4]**

Convexity:

Modified duration is also not an efficient to predict the change in price of a bond with a given change in the yield, since, as we noted earlier, the relation between yield and prices is curvilinear. As the curve is convex from below, when the yield falls by Δy, the resulting increase in price is not just $D^*_m \times \Delta y$, but more. When the yield increases by Δy, the resulting decrease in price is not $D^*_m \times \Delta y$, but less. Hence, convexity of the curve increases the increase in price, and reduces the loss in price, compared to what is given by a linear measure of duration. Hence, quite often, investors see the convexity as a gain, and the realized increase/decrease in price over linear duration measure is referred to as convexity gain.

Convexity is the second differential in the price/yield relationship. Convexity may be computing by differentiating Duration 2 with respect to y once again, as follows:

$$D_m^* = \frac{\partial p}{\partial y} = -\frac{D_m}{(1+y)} = -\sum_{t=1}^{n} \frac{C_t t}{(1+y)^{t+1}}$$ ...Duration 2

Now, differentiating w.r.t y, we have

$$\frac{\partial D_m^*}{\partial y} = \sum_{t=1}^{n} \frac{C_t . t(t+1)}{(1+y)^{t+2}}$$ [**Duration 5**]

Predicting change in price on changes in yields:

In any curvilinear function such as the price/yield function, the change in price of a bond with a given change in yield (Δy) is computed as follows:

$\Delta p = p + D^*_m \times \Delta y + ½ C \Delta y^2$ [**Duration 6**]

This equation is obtained by Taylor's series expansion method as follows :

Consider a dependent variable 'y' which is a function of the independent variable 'x' i.e.

y=f(x). Now we ask the question: if there is a change in the value of 'x' (say 'x' changes to 'x+Δx') what will be the corresponding change in the value of 'y' i.e. f(x+Δx)-f(x)=?.When the function 'f(x)' satisfies some special conditions then Taylor's Theorem provides us with an answer in the form of the following result:

$$f(x+\Delta x) = f(x) + f^1(x)\Delta x + \frac{f^2(x)(\Delta x)^2}{2!} + \cdots + \frac{f^n(x)(\Delta x)^n}{n!} + \cdots$$

where $f^n(x) = the\ nth\ orderderivative\ of\ \ 'f'\ at\ \ 'x'.$

Luckily, the function at hand does satisfy these conditions.Now any small change in y, the third and the subsequent terms in the equation above become insignificant, and may be ignored. Hence, we get Duration 6.

$$p = p + D * m x y + ½ C y2$$