

# Capital asset pricing model, arbitrage pricing theory and portfolio management

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The capital asset pricing model (CAPM) is great in terms of its understanding of risk – decomposition of risk into security-specific risk and market risk. Before we discuss the CAPM, it would be important to understand risk of portfolios.

## ***Risk of portfolios:***

Markowitz made a seminal contribution to theory of investments way back in 1959 when he propounded his efficient portfolio theory. Markowitz contended, and established with basic statistics, that when a risky security is combined with another risky securities, unless the two are closely correlated, the risk of the portfolio (that is, the two securities put together) does not go up – it comes down.

To understand this point, let us first understand the risk of securities, and then we move on to risk of portfolios.

## **Risk of securities:**

Let us take an example.

### **Example 1**

Let us suppose there are two securities, with the following returns profile:

Scenario	Security A	Probability	Security B	Probability
1	10%	10%	5%	10%
2	11%	20%	11%	20%
3	12%	40%	15%	40%
4	13%	20%	19%	20%
5	14%	10%	25%	10%
Expected returns		12.00%		15.00%

The expected returns in the last row have been computed by multiplying the returns in each scenario, by the probability of the scenario, and adding up the products. That is to say:

$$\bar{R} = \sum_{i=1}^n R_i P_i \quad (1)$$

where:

$\bar{R}$  : the expected returns from the portfolio

$R_i$  : the returns in scenario i

$P_i$ : the probability of scenario i

It is quite clear that Security B has higher expected returns than Security A. However, it is also apparent, even to naked eye, that the variability of returns of Security B is substantially higher. The variability of returns is captured by computing the standard deviation of the two securities, which we do below:

Scenario	Security A	Probability	$(\overline{R} - R)^2$	$((\overline{R} - R)^2) * P$	Security B	Probability	$(\overline{R} - R)^2$	$((\overline{R} - R)^2) * P$
1	2	3	4	5	6	7	8	9
1	10%	10%	0.0004	0.00004	5%	10%	0.01	0.001
2	11%	20%	0.0001	0.00002	11%	20%	0.0016	0.00032
3	12%	40%	0	0	15%	40%	0	0
4	13%	20%	0.0001	0.00002	19%	20%	0.0016	0.00032
5	14%	10%	0.0004	0.00004	25%	10%	0.01	0.001
			0.001	0.00012				
Expected returns		12.00%				15.00%		
Standard deviation		1.095%				5.138%		

The standard deviation has been computed by adding up Col 5/ Col 9 respectively, and taking the root of the same.

$$\sigma = \sqrt{\sum (R - \overline{R})^2 P} \quad (2)$$

### Making a portfolio:

We have so far seen the individual returns of the two securities. An investor has a particular sum of money to invest, which he may invest either entirely in Security A, or entirely in Security B, or he may hold various combinations of A and B. Let us suppose these options are summed in the following table:

Composition of Portfolio	Security A	Security B	Portfolio returns
1	2	3	4
1	100%	0%	12.00%
2	90.0%	10%	12.30%
3	80.0%	20%	12.60%
4	70.0%	30%	12.90%
5	60.0%	40%	13.20%
6	50.0%	50%	13.50%
7	40.0%	60%	13.80%
8	30.0%	70%	14.10%
9	20.0%	80%	14.40%
10	10.0%	90%	14.70%
11	0.0%	100%	15.00%

There might be infinite ways of combining Security A and B, but we have taken above 11 scenarios, changing the weight of the two securities from 100% of Security A, sliding it

down to 0%. The expected returns of the portfolio, in Col 4 above, is simply the returns from Security A and Security B, weighted in their respective proportions as given by Col 2 and 3. That is to say,

$$R_p = R_A \cdot X_A + R_B \cdot X_B$$

Or generalizing:

$$R_p = \sum_{i=1}^n R_i \cdot X_i \quad (3)$$

Where

$R_p$  : Expected return from the Portfolio

$X_i$  : Proportion of security i in the portfolio

Needless to say, as we add more of Security B to the portfolio, the returns from the portfolio continue to go up. This is quite obvious, as B gives higher returns.

But then B has higher risk too. Does that mean, as B is added to the portfolio, the risk of the portfolio also goes up? That is exactly where Markowitz made a significant point, holding that as doses of Security B are added to Security A, while the expected return goes up, the risk does not go up, at least upto a particular level.

### Portfolio risk:

The risk or the standard deviation of the portfolio is given by:

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n X_i X_j \sigma_{ij} \quad (4)$$

and

$$\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$$

where  $X_i$ ,  $X_j$  etc are the proportions of the respective assets in the portfolio.  $\rho_{ij}$  is the correlation between i-th asset and j-th asset.  $\sigma_{ij}$  is also referred to as the co-variance of i-th and j-th security.

Needless to say, to get the  $\sigma_p$  from Eq 4, all we have to do it to take its square root.

The above formula is intuitively understandable. As in case of the mean, the standard deviation is also the weighted average of the standard deviations of the two securities; however, it is the correlation that is making a difference here. If the correlation is 1, the covariance is the same as the weighted average of the standard deviations. However, where correlation is less than 1, it causes the covariance of the portfolio to come down.

### Example 2

Let us assume we have two securities which have the following risk return profile

	Security A	Security B
Mean returns	10%	15%
Standard deviation	3%	6%

Let us assume that the correlation between the returns of the two securities is 0.25 or 25%.

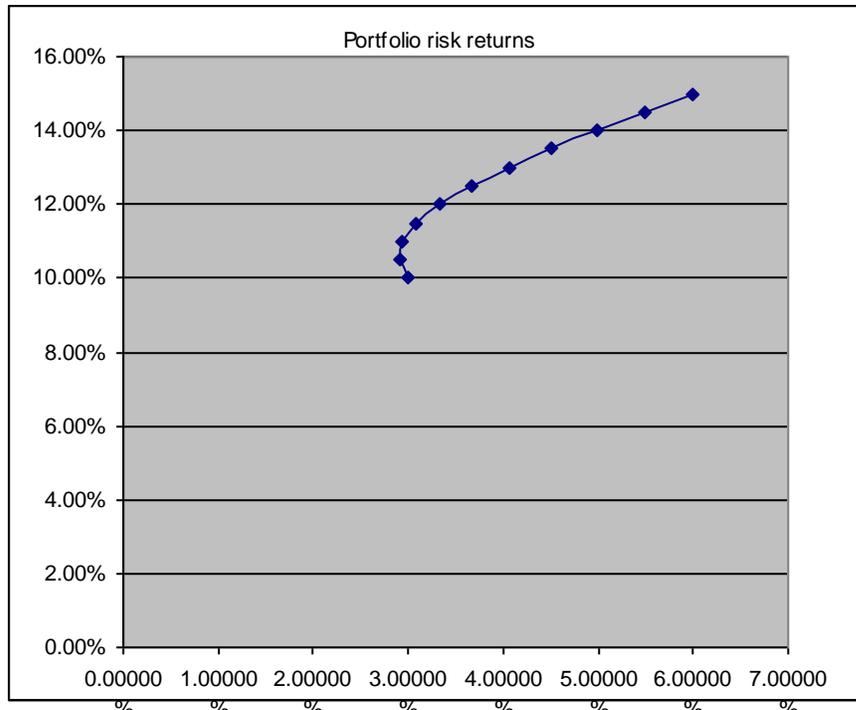
We the above realignment, the correlation between Security A and Security B is approximately 25%.

Now, if we put a correlation assumption of 25%, and compute portfolio risk as per Eq. 4, let us say, with 90% of Stock A and 10% of Stock B, we get a portfolio standard deviation of 2.90861%, which is less than 3%, the standard deviation of security A only. This may, at first sight seem a little strange – we added a riskier security (B), and yet, the combined result is less risky than the single security. It is like mixing chilly with sugar, and the result being sweeter! However, on further reflection, it is not difficult to understand this – the correlation between the two securities is low. With the lower degree of correlation acts as a risk absorption device – the risk comes down even though, adding Security B to the portfolio, the returns go up.

With different combinations of Security A and Security B, keeping correlation of 0.25, the risks/returns look as follows:

Proportion of A	Proportion of B	Portfolio SD	Portfolio returns
1	0	3.00000%	10.00%
0.9	0.1	2.90861%	10.50%
0.8	0.2	2.93939%	11.00%
0.7	0.3	3.08869%	11.50%
0.6	0.4	3.34066%	12.00%
0.5	0.5	3.67423%	12.50%
0.4	0.6	4.06940%	13.00%
0.3	0.7	4.50999%	13.50%
0.2	0.8	4.98397%	14.00%
0.1	0.9	5.48270%	14.50%
0	1	6.00000%	15.00%

If we were to plot these results on a graph (risk on X axis and returns on Y axis), the graph looks like the one below:



Note the bulge of the graph towards the left – this indicates the reduction in risk with increasing returns upto a point, beyond which the risk starts increasing with increasing returns. The least risky position for an investor is the left-most point on the bulging curve. However, this point need not necessarily be the ideal choice for the investor, as the investor may, indeed, be comfortable with a higher dose of risk, but with increased returns.

It would not be difficult to understand that as the correlation between the two securities is increased, the bulging curve starts getting flatter.

The Table below shows the portfolio standard deviation for different correlation levels:

Proportion of A	Correlation between A and B				
	0.25	0.4	0.5	0.8	1
100.00%	3.0000%	3.0000%	3.0000%	3.0000%	3.0000%
90.00%	2.9086%	2.9910%	3.0447%	3.2003%	3.3000%
80.00%	2.9394%	3.0829%	3.1749%	3.4363%	3.6000%
70.00%	3.0887%	3.2671%	3.3808%	3.7011%	3.9000%
60.00%	3.3407%	3.5293%	3.6497%	3.9890%	4.2000%
50.00%	3.6742%	3.8536%	3.9686%	4.2953%	4.5000%
40.00%	4.0694%	4.2256%	4.3267%	4.6165%	4.8000%
30.00%	4.5100%	4.6340%	4.7149%	4.9495%	5.1000%
20.00%	4.9840%	5.0699%	5.1264%	5.2923%	5.4000%
10.00%	5.4827%	5.5268%	5.5561%	5.6429%	5.7000%
0.00%	6.0000%	6.0000%	6.0000%	6.0000%	6.0000%

## CAPM model

- The risk of the market dictates the returns from the market – as the risk goes up, the returns of the market also go up. This relationship is given by the **capital market line**.
- As for an individual security, the relationship between the market returns ( $R_m$ ) and the returns from the individual security  $j$  ( $R_j$ ) depends on the sensitivity of  $R_j$  with  $R_m$ . The slope of the line that relates  $R_j$  with  $R_m$  is called the **beta** of security  $j$ .
- If the beta is more than 1, the security is more sensitive
- Generalized formula for  $R_j$ :

$$R_j = R_f + \text{beta} (R_m - R_f)$$

Where

$R_j$  - expected return on security  $j$

$R_f$  - risk free rate of return

$R_m$  - market returns

- Generalized formula for beta: covariance of security  $j$  with market divided by variance of the market

$$\text{Beta} = (\sigma_j \sigma_m \rho_{jm}) / \sigma_m^2$$

- Placing equation 2 in equation 1, we have the following result:
  - The spread provided by the market is a function of the deviation of the market. This spread, divided by the market deviation, multiplied by deviation of security  $j$ , multiplied by its correlation, provides the spread given by security  $j$
- The risk introduced by the beta is risk derived from the market – this risk is, therefore, the **market risk** or **systematic risk**. This risk is non-diversifiable.
- The actual return on a particular security will include the error term, that is, the deviation between the realized return and the expected return. This error term is subject to reduction by diversification. In other words, this risk is **diversifiable risk**, also called **idiosyncratic risk** or **unsystematic risk**.

## Arbitrage Pricing Theory (APT)

The essence of the CAPM was that the pricing of the individual security in the market is done based on its beta, that is, sensitivity of the stock to the market returns. In the actual realized returns, there will be a difference to the idiosyncratic error term  $e$ , but the expected value of  $e$  equals zero. At the same time,  $e$  is diversifiable. Hence, the CAPM

believes the only factor that affects returns from the particular security is the market return.

APT has been propounded by Ross. APT is also an equilibrium model – explaining that the market process brings security prices ultimately at an equilibrium.

The arbitrage pricing theory seeks to explain the process of pricing of securities in the market as the process whereby investors try and exploit arbitrage opportunities until arbitrage opportunities are completely killed. That is, the market reaches an equilibrium when there is no opportunity to make arbitrage or riskless profit. Where do we say there is an arbitrage opportunity? When do we say a stock is too cheap or too overpriced? Analysts try and identify factors that explain market prices, and then correlate the movement of those factors with the movement of the prices. There might be several such factors explaining market prices: the factors might be macroeconomic, or industry-specific. In other words, the factor model sees a linear relationship between variables that explain the market prices, and sensitivity of changes in each of these variables with the changes in the security prices. These variables or **factors** may also be perceived as the risks that affect the price of the security.

In the CAPM, the only model that explained changes in the security prices/returns was the market return. In the APT, the factors may be several. Hence, CAPM was a single-factor model; the APT is a multi-factor model.

In the APT model, the price of the security is explained as follows:

**Two factor equation:**

$$R_j = a + b_{1j} F_1 + b_{2j} F_2 + e_j \quad (\text{APT1})$$

**Multi factor model:**

$$R_j = a + b_{1j} F_1 + b_{2j} F_2 + \dots + b_{mj} F_m + e_j \quad (\text{APT2})$$

Let us try understanding APT1. The first term  $a$  is the return when the causative factors or risks have zero value. This may be perceived as the risk-free rate.  $F_1$  and  $F_2$  are factors that affect the prices of security  $j$ , and  $B_1$  and  $B_2$  are degree to which the factors affect the returns from the security, that is, the sensitivity of the returns from security  $j$  to the respective factors. These are the betas. There are separate betas for each of the factors – hence, we have  $b_{1j}$ ,  $b_{2j}$  and so on. The last term in the equation is the error term or the variability of the realized return from the return explained by the betas and the factors. As in case of the CAPM, the expected value of  $e$  will be zero – that is, it will have gains and losses that will neutralize.

The extension to multi-factor model (Eq APT-2) is not very difficult. We have simply extended the equation to include multiple factors with their respective betas.

In APT, there are as many betas as there are factors that affect the price of the security.

Yet another way to understand the APT would be to look at the pricing of a security as composed of risk free rate, and risk premiums representing different risks. The factors may also be perceived as different risk premiums, with the betas being the multipliers for these different risk premiums.

Why is it called arbitrage pricing? If the betas of two different securities with a given factor F1 are known, then the expected

## Sharpe Index model

William Sharpe's Sharpe Index model or single index model is actually a precursor to the CAPM, and is a simplification of Markowitz. Under the Markowitz model, the risk of the portfolio is affected by covariance of pairs of securities – if we were to extend the formula for risk under Markowitz below, there will  $n * (n-1)/2$  co-variances.

$$\sigma^2_p = \sum_{i=1}^n \sum_{j=1}^n X_i X_j \sigma_{ij}$$

Instead, Sharpe suggested that the required inputs under the Markowitz model may be simplified by looking at the correlation of the security with a broader market index, instead of pairs of securities. Hence, the return of the individual security can be seen as:

$$R_j = a + b_j R_m + e_j$$

Where  $R_m$  is return from the market, and the beta is the sensitivity of the stock to the market.